GEOMORPHOMETRY 2020 PERUGIA, ITALY JUNE 22 - 26 2020

The surface stream function: representing flow topology with numbers

John Gallant[§]
CSIRO Land and Water
Black Mountain Laboratories
Clunies Ross St, Canberra ACT, Australia
[§] john.gallant@csiro.au

Abstract— **This paper introduces the concept of the land surface stream function, a map that labels slope lines in a way that captures the topology of surface flow networks. When combined with elevation, the stream function forms a hydrological, rather than geographical, coordinate system that reflects the organisation of the landscape into nested catchments with hierarchical stream and ridge networks.**

The stream function map dramatically simplifies several otherwise complicated tasks related to the analysis of surface flow paths. It identifies a catchment at any scale with a pair of numbers, the minimum and maximum stream function for the catchment. It supports tracing of catchment boundaries from any point without needing to follow flow pathways within the catchment, and it has the potential to dramatically simplify the construction of surface flow nets for hydrological modelling.

I. Introduction

Flow across the landscape can be visualized with the aid of contour and slope lines. The shapes of the contour lines convey information about the form of the land surface and show specific features such as ridges, valleys, hilltops, saddles and depressions. Their spacing indicates the slope of the surface. Slope lines follow the direction of steepest descent and represent the paths of water flow across the land surface [1,2]. Slope lines are everywhere perpendicular to contour lines and suitably chosen slope lines divide the landscape into hydrological units of catchments and interfluves. In combination with the contour lines they create an orthogonal curvilinear mesh that can be used to partition the land surface into broad units (Maxwell's Hills and Dales) or more finely into stream tubes and elements for modelling surface flow [3-7]. Slope lines are also the basis for the mathematical formulation of specific catchment area [8].

Contour lines represent lines of constant elevation, but what quantity can be associated with a slope line? The surface stream function assigns values to slope lines so that the difference in value from one slope line to another equals the area between the two slope lines. With the addition of the stream function field, the orthogonal curvilinear mesh of contours and slope lines becomes a physically meaningful coordinate system. It is closely related to the stream function used in groundwater and other fluid flow applications, with the contour lines representing the potential and the slope lines representing the flux but using area rather than an actual material flow; this matches the way specific catchment area is used as a surrogate for flow in surface hydrology.

II. The Stream Function Concept

A. Definitions

The properties of the stream function are intimately connected with the properties of slope lines. Every slope line runs from a peak to a sink, except for particular lines that terminate on a saddle at either the upper or lower end. Maxwell [1] envisaged slope lines running to underwater pits but in practice slope lines are considered to terminate at water bodies (lakes or the coastline). The lines that terminate on a saddle play important roles so are here given specific names:

- A **course line** runs from a saddle to a sink (or coastline)
- A **divide line** runs from a peak to a saddle

The divide and course lines are important *topological* features in the slope line network and the names are chosen to avoid using the names valley and ridge, which are here reserved for *topographic* features. Course lines often run along valleys but not always; likewise divide lines are often found on ridges but not always. In terms of slope lines, these definitions of valley and ridge are adopted here:

John Gallant (2020) The surface stream function: representing flow topology with numbers:

in Massimiliano Alvioli, Ivan Marchesini, Laura Melelli & Peter Guth, eds., Proceedings of the Geomorphometry 2020 Conference, doi:10.30437/GEOMORPHOMETRY2020_60.

- A **valley line** is a slope line to which many slope lines converge in a downslope direction heading towards a sink (or coastline)
- A **ridge line** is a slope line to which many slope lines converge in an upslope direction heading towards a peak

B. Stream function fundamentals

Each slope line has a unique stream function value not shared with any other line. Unlike contours there cannot be multiple disconnected line segments with the same value. The only exception to this is slope lines terminating on saddles where multiple slope lines share a stream function value as shown in the examples below, but even then the multiple line segments all touch at a single saddle.

In general, a point on the landscape has only one slope line running through it so has a single stream function value, but there are important exceptions. Because multiple slope lines connect to each peak, peaks have a range of stream function values. Likewise sinks have multiple slope lines and a range of stream function values. Ridge and valley lines are in practice considered to have a range of stream function values that varies along the line.

The basic relationship between stream function value, denoted by ψ, and conventional ideas of stream tube area and contributing and dispersal areas A and D is that the total area between the two slope lines (the stream tube area, see Figure 1) equals the change in stream function value between the lines:

$$\Delta\psi = \psi_2 - \psi_1 = A + D \tag{1}$$

Stream function is defined here such that ψ increases from left to right when looking downslope. Hence ψ increases when moving along a contour line in a clockwise direction around a peak and in a counter-clockwise direction around a sink. Note that this makes the (ψ, z) coordinate system left-handed (Figure 1).

C. Examples of stream function properties

Figure 2a shows an idealized small island with an area of 100 ha and a single peak in the centre at a height of 100 m. Slope lines are arranged around the island enclosing areas of 10 ha between each pair of lines so the stream function increases by 10 ha from one line to the next.

The location of the slope line with $\psi = 0$ is arbitrary, as only the differences in stream function value have any significance. The stream function increases clockwise around the island and reaches a maximum of 100, the island's area, on the other side of the starting slope line. Every contiguous land mass will have one of these lines of discontinuity where the stream function jumps by the total area of the land mass.

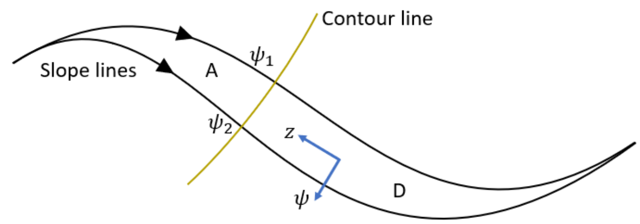

Figure 1. Plan view of a stream tube bounded by two nearby slope lines that terminate or merge at each end. Each slope line has an associated stream function value ψ. At any contour line along the stream tube, the total area of the stream tube is divided into the contributing area A above the contour line and the dispersal area D below the line. ψ and z form a left-handed curvilinear coordinate system.

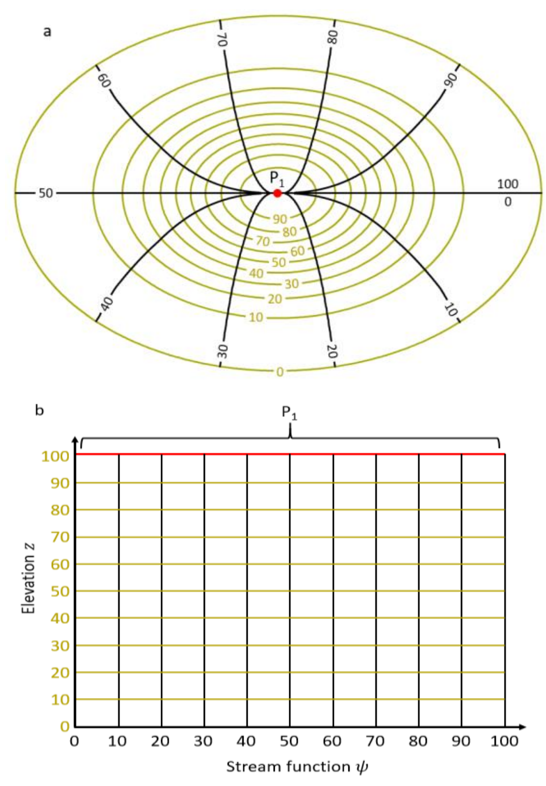

Figure 2. (a) An island of 100 ha area with a single peak P_1 at 100 m elevation. (b) The stream function map for the island showing contour and slope lines parallel to the z and ψ axes.

Figure 2b shows the same island in the hydrological (ψ, z) space. In this representation the contour lines are horizontal since they all have constant z values and the slope lines are vertical with constant ψ values. The peak P_1 appears as a horizontal line since all slope lines connect to it so it includes the full range of stream function values corresponding to the peak's dispersal area.

Figure 3a shows a similar island but with two peaks, P_1 and P_2 at heights of 70 and 100 m. Between the peaks there is a saddle K_1 at a height of about 45 m and the contour at the saddle elevation forms two closed loops in a figure-eight formation. The two peaks P_1 and P_2 still appear as horizontal lines since all the slope lines have their upper end at one of those two peaks.

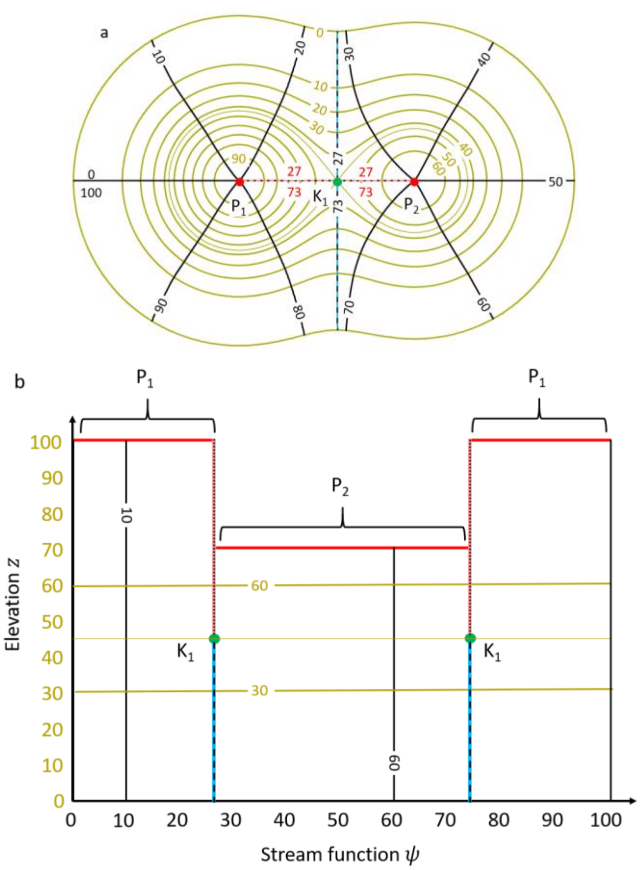

Figure 3. (a) An island with two peaks, P_1 and P_2 at elevations 100 and 70 respectively and an intervening saddle K_1 at an elevation of about 45. Divide lines (red with short black dashes) run from each peak to the saddle and course lines (blue with long black dashes) run from the saddle to both sides of the island. (b) The stream function map for the island showing the saddle K_1 with its two different stream function values. The labelled contour and slope lines correspond to the same labels in (a).

The saddle K_1 introduces significant changes to the topology of the flow lines and therefore changes in the structure of the island in (ψ, z) space. The fact that the saddle point K_1 is connected to two course lines that terminate on opposite sides of the island means that the saddle point has two different stream function values, 27 and 73 (note that integer values of ψ are used here for convenience, in reality they are real values). As a result, the saddle point K_1 appears twice in the (ψ, z) map with the same z but different ψ. The contour line through the saddle appears as a single line in the (ψ, z) map but still crosses K_1 twice because K_1 appears twice in that map. The contour lines at elevations 30 and 60 in the (ψ, z) map appear to both be simple lines, even though the contour at 60 m is two separate loops in (x, y) space. The discontinuity in stream function along the divide lines in (x, y) space is matched by discontinuities in x and y on the divide lines in (ψ, z) space.

The divide lines running from the peaks to the saddle have two stream function values, the same values as the two course lines, reflecting the fact that those slope lines effectively continue past the saddle as course lines to reach the coastline on opposite sides of the island. The dual values are valid on opposite sides of the lines and the values between them are not included: there is no slope line with $\psi = 50$ reaching peak P_1, for example. Divide lines create a discontinuity in the stream function.

Sinks behave in a complementary manner to peaks – they appear in (ψ, z) space as a horizontal line with a stream function range equal to their contributing area. Course lines from saddles connected to sinks have a discontinuity in stream function value.

D. Stream function in a real catchment

Figure 4 shows the geographic and stream function map for a 436 ha catchment in Australia (150° 42′ E 33°16′ S). Ridges and valleys, defined using a threshold area of 10 ha, are identified as thicker red and blue lines in the geographic map and as shaded red and blue areas in the stream function map. Much of the stream function map is occupied by valleys (blue regions) reflecting the fact that most slope lines run for most of their height along a valley line, which will usually be channelled. The branched valley network produces a set of nested truncated triangular regions in the (ψ, z) map. The sloped edges of those regions capture the changing range of ψ along the valley lines.

I. APPLICATIONS

Because stream function captures flow topology in a numerical system it replaces many otherwise complex flow-tracing operations with much simpler operations. The examples here assume stream function is implemented on a raster where each

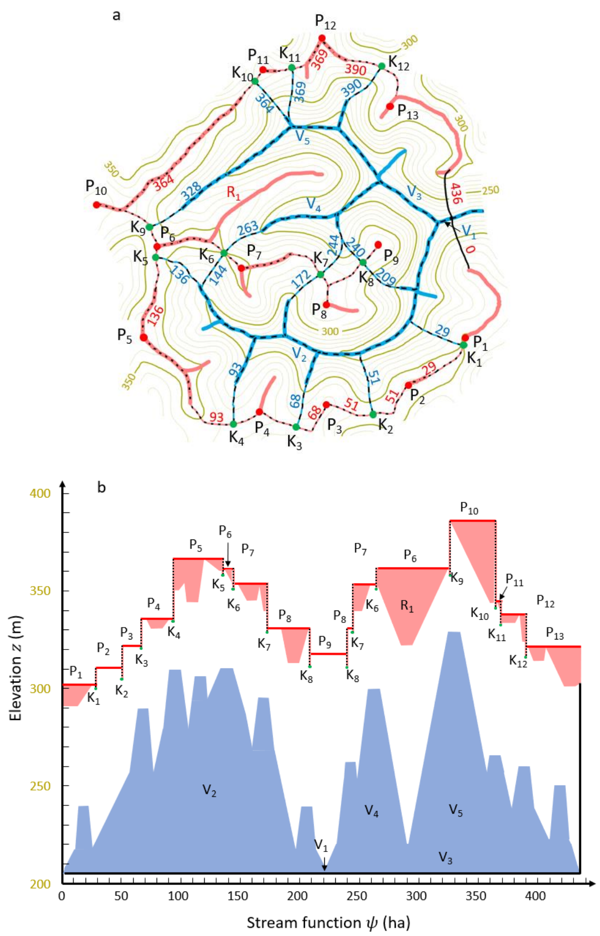

Figure 4. (a) A small catchment with branched ridge and valley networks. (b) The stream function map for the catchment. The blue and red shaded areas correspond to the blue valley lines and red ridge lines in (a).

cell has both a minimum and maximum value of stream function. The difference between the minimum and maximum is the sum of contributing area and dispersal area for the cell, as in Figure 1.

A. Catchments and partial catchments

Stream function defines the catchment of an outlet cell as all the cells satisfying:

$$\psi_{min} \geq \psi_{min,outlet} \text{ and } \psi_{max} \leq \psi_{max,outlet}$$

That inequality can be used to find all cells in a catchment or decide if a single location lies within a given catchment. It is also possible to trace the outline of a catchment using stream function without visiting any of the interior cells.

Along a stream reach from point A to point B, the area on the left side of the stream (looking downstream) has:

$$\psi_{min,A} \leq \psi_{min} \leq \psi_{min,B}$$

while the area on the right side of the stream has:

$$\psi_{max,B} \leq \psi_{max} \leq \psi_{max,A}$$

B. Flow nets for hydrological modelling

The original motivation for developing the stream function map was to simplify the creation of flow nets – all previous methods for producing them are based on contour data. Elements in a flow net are bounded by contour lines and slope lines; in the stream function map, these are just rectangles in (ψ, z) space. Despite this simplicity there are still complexities to overcome, particularly in the automated division of the landscape into elements of roughly uniform size, and the implementation is yet to be completed.

ACKNOWLEDGEMENTS

The development of the stream function concept and its implementations has taken many years. Early steps were supported by the CRC for Catchment Hydrology and Professor Joe Ritchie at Michigan State University. The bulk of the development of the Terrae software implementation was funded by Professor Bruno Basso at University of Basilicata and Michigan State University, and most of the details of the behaviour of stream function were worked out during that period. Prof Michael Hutchinson provided valuable insights at several key points during the development of the stream function methods.

REFERENCES

[1] Cayley, A. 1859. "On contour and slope lines" The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science 18(120): 264-268.

[2] Maxwell, J.C. 1870. "On hills and dales" The London Edinburgh and Dublin Philosophical Magazine and Journal of Science 40: 421-427.

[3] Onstad, C.A. and Brakensiek, D.L. 1968. "Watershed simulation by the stream path analogy" Water Resources Research 4(5): 965-971.

[4] O'Loughlin, E.M. 1986. "Prediction of surface saturation zones in natural catchments by topographic analysis." Water Resources Research 22(5): 794-804.

[5] Moore, I.D. and Grayson, R.B. 1991. "Terrain-based catchment partitioning and runoff prediction using vector elevation data." Water Resources Research 27(6): 1177-1191.

[6] Dawes, W. R. and Short, D. 1994. "The significance of topology for modeling the surface hydrology of fluvial landscapes." Water Resources Research 30(4): 1045-1055.

[7] Moretti, G. and Orlandini, S. 2008. "Automatic delineation of drainage basins from contour elevation data using skeleton construction techniques." Water Resources Research 44: W05403.

[8] Gallant, J.C. and Hutchinson, M.F. 2011. "A differential equation for specific catchment area." Water Resources Research 47: W05535.